Broken Replica Symmetry Bounds in the Mean Field Spin Glass Model

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Abstract

By using a simple interpolation argument, in previous work we have proven the existence of the thermodynamic limit, for mean field disordered models, including the Sherrington-Kirkpatrick model, and the Derrida p-spin model. Here we extend this argument in order to compare the limiting free energy with the expression given by the Parisi *Ansatz*, and including full spontaneous replica symmetry breaking. Our main result is that the quenched average of the free energy is bounded from below by the value given in the Parisi *Ansatz*, uniformly in the size of the system. Moreover, the difference between the two expressions is given in the form of a sum rule, extending our previous work on the comparison between the true free energy and its replica symmetric Sherrington-Kirkpatrick approximation. We give also a variational bound for the infinite volume limit of the ground state energy per site.

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1 Introduction

The main objective of this paper is to compare the free energy of the mean field spin glass model, introduced by Sherrington and Kirkpatrick in [16], with the expression given in the frame of the Parisi *Ansatz* [14], [12], including the complete phenomenon of spontaneous replica symmetry breaking. In previous work [6], we have limited our comparison to the replica symmetric case, by producing sum rules, where the difference between the true free energy, and its replica symmetric approximation, is expressed in terms of overlap fluctuations, with a well definite sign. As a result, the replica symmetric approximation turns out to be a rigorous lower bound for the quenched average of the free energy per site, uniformly in the size of the system.

In the meantime, the old problem of proving the existence of the infinite volume limit for the thermodynamic quantities has been solved [9], by using a simple comparison argument.

Here, we extend this comparison argument, by introducing an appropriate generalized partition function, as a function of a parameter t, with $0 \leq t \leq 1$, able to interpolate between the true theory, at $t = 1$, and the broken replica *Ansatz*, at $t = 0$. Consequently, through a simple direct calculation, we can evaluate the difference between the true free energy, and its broken replica expression, still in the form of a sum rule, with the corrections, of a definite sign, expressed through overlap fluctuations, in properly chosen auxiliary states. As a result, the broken replica *Ansatz* turns out to be a rigorous lower bound for the quenched average of the free energy per site, uniformly in the size of the system.

Moreover, the corrections, given in terms of overlap fluctuations, are in a form suitable for the exploration of the expected result of their vanishing, when the size of the system goes to infinite, along the program developed in [8].

Of course, our method does not use the replica trick in the zero replica limit, as explained for example in [12], nor the cavity method, as exploited for example in [13], [15], [5], [17].

We give only a brief sketch of the extension of our method to the Derrida p-spin model [2], [4], [3], [17]. A more detailed treatment will be presented elsewhere [10].

The organization of the paper is as follows. In Section 2, we will briefly recall the main features, and definitions, of the mean field spin glass model. In Section 3, the general structure of the Parisi spontaneously broken replica symmetry *Ansatz* will be described, in a form suitable for the developments of next Section. Section 4 contains the main results of the paper. Firstly, we introduce the interpolating generalized partition function. Then, we evaluate its derivative, with respect to the interpolating parameter, arriving to the sum rule. The general broken replica bound follows easily. In Section 5, we give a variational estimate for the infinite volume limit of the ground state enrgy per site. Next Section 6 gives the essential ingredients of the extension of this method to the p-spin model. Finally, Section 7 is devoted to conclusions and outlook for further developments.

2 The basic definitions for the mean field spin glass model

The generic configuration of the mean field spin glass model is defined through Ising spin variables $\sigma_i = \pm 1$, attached to each site $i = 1, 2, \ldots, N$.

The external quenched disorder is given by the $N(N-1)/2$ independent and identical distributed random variables J_{ij}, defined for each couple of sites. For the sake of simplicity, we assume each J_{ij} to be a centered unit Gaussian with averages $E(J_{ij}) = 0$, $E(J_{ij}^2) = 1$.

The Hamiltonian of the model, in some external field of strength h, is given by the mean field expression

$$H_N(\sigma, h, J) = -\frac{1}{\sqrt{N}} \sum_{(i,j)} J_{ij}\sigma_i\sigma_j - h\sum_i \sigma_i. \tag{1}$$

Here, the first sum extends to all site couples, an the second to all sites.

For a given inverse temperature β, let us now introduce the disorder dependent partition function $Z_N(\beta, h, J)$, the quenched average of the free energy per site $f_N(\beta, h)$, the Boltzmann state ω_J, and the auxiliary function $\alpha_N(\beta, h)$, according to the well known definitions

$$Z_N(\beta, h, J) = \sum_{\sigma_1\ldots\sigma_N} \exp(-\beta H_N(\sigma, h, J)), \tag{2}$$

$$-\beta f_N(\beta, h) = N^{-1} E \log Z_N(\beta, h, J) = \alpha_N(\beta, h), \tag{3}$$

$$\omega_J(A) = Z_N(\beta, h, J)^{-1} \sum_{\sigma_1\ldots\sigma_N} A \exp(-\beta H_N(\sigma, h, J)), \tag{4}$$

where A is a generic function of the σ's.

Replicas are introduced by considering a generic number s of independent copies of the system, characterized by the Boltzmann variables $\sigma_i^{(1)}$, $\sigma_i^{(2)}$, $\ldots$, distributed according to the product state $\Omega_J = \omega_J^{(1)}\omega_J^{(2)}\ldots\omega_J^{(s)}$. Here, all

$\omega_J^{(\alpha)}$ act on each one $\sigma_i^{(\alpha)}$'s, and are subject to the *same* sample J of the external noise.

The overlap between two replicas a, b is defined according to $q_{ab} = N^{-1} \sum_i \sigma_i^{(a)} \sigma_i^{(b)}$, with the obvious bounds $-1 \leq q_{ab} \leq 1$.

For a generic smooth function F of the overlaps, we define the $\langle \ \rangle$ averages

$$\langle F(q_{12}, q_{13}, \ldots) \rangle = E\Omega_J(F(q_{12}, q_{13}, \ldots)), \tag{5}$$

where the Boltzmann averages Ω_J act on the replicated σ variables, and E is the average with respect to the external noise J.

3 The broken replica symmetry *Ansatz*

While we refer to the original paper [14], and to the extensive review given in [12], for the general motivations, and the derivation of the broken replica *Ansatz*, in the frame of the ingenious replica trick, here we limit ourselves to a synthetic description of its general structure, in a form suitable for the treatment of next Section, see also [5], [1].

First of all, let us introduce the convex space $\mathcal{X}$ of the functional order parameters x, as nondecreasing functions of the auxiliary variable q, both x and q taking values on the interval $[0, 1]$, *i.e.*

$$\mathcal{X} \ni x : [0, 1] \ni q \to x(q) \in [0, 1]. \tag{6}$$

Notice that we call x the nondecreasing function, and $x(q)$ its values. We introduce a metric on $\mathcal{X}$ through the $L^1([0, 1], dq)$ norm, where dq is the Lebesgue measure.

Usually, we will consider the case of piecewise constant functional order parameters, characterized by an integer K, and two sequences $q_0, q_1, \ldots, q_K$, $m_1, m_2, \ldots, m_K$ of numbers satisfying

$$0 = q_0 \leq q_1 \leq \ldots \leq q_{K-1} \leq q_K = 1, \ \ 0 \leq m_1 \leq m_2 \leq \ldots \leq m_K \leq 1, \tag{7}$$

such that

$$\begin{aligned} x(q) = m_1 \text{ for } 0 = q_0 \leq q < q_1, \ \ x(q) = m_2 \quad &\text{for} \quad q_1 \leq q < q_2, \\ \ldots, x(q) = m_K \quad &\text{for} \quad q_{K-1} \leq q \leq q_K. \end{aligned} \tag{8}$$

In the following, we will find convenient to define also $m_0 \equiv 0$, and $m_{K+1} \equiv 1$. The replica symmetric case corresponds to

$$K = 2, \ \ q_1 = \bar{q}, \ \ m_1 = 0, \ \ m_2 = 1. \tag{9}$$

The case $K = 3$ is the first level of replica symmetry breaking, and so on.

Let us now introduce the function f, with values $f(q, y; x, \beta)$, of the variables $q \in [0, 1]$, $y \in R$, depending also on the functional order parameter x, and on the inverse temperature β, defined as the solution of the nonlinear antiparabolic equation

$$(\partial_q f)(q, y) + \frac{1}{2}(f''(q, y) + x(q) f'^2(q, y)) = 0, \tag{10}$$

with final condition

$$f(1, y) = \log \cosh(\beta y). \tag{11}$$

Here, we have stressed only the dependence of f on q and y, and have put $f' = \partial_y f$ and $f'' = \partial_y^2 f$.

It is very simple to integrate Eq. (10) when x is piecewise constant. In fact, consider $x(q) = m_a$, for $q_{a-1} \le q \le q_a$, firstly with $m_a > 0$. Then, it is immediately seen that the correct solution of Eq. (10) in this interval, with the right final boundary condition at $q = q_a$, is given by

$$f(q, y) = \frac{1}{m_a} \log \int \exp(m_a f(q_a, y + z\sqrt{q_a - q}))\, d\mu(z), \tag{12}$$

where $d\mu(z)$ is the centered unit Gaussian measure on the real line. On the other hand, if $m_a = 0$, then (10) loses the nonlinear part and the solution is given by

$$f(q, y) = \int f(q_a, y + z\sqrt{q_a - q})\, d\mu(z), \tag{13}$$

which can be seen also as deriving from (12) in the limit $m_a \to 0$. Starting from the last interval K, and using (12) iteratively on each interval, we easily get the solution of (10), (11), in the case of piecewise order parameter x, as in (8).

We refer to [7] for a detailed discussion about the properties of the solution $f(q, y; x, \beta)$ of the antiparabolic equation (10), with final condition (11), as a functional of a generic given x, as in (8). Here we only state the following

Theorem 1. *The function f is monotone in x, in the sense that $x(q) \le \bar{x}(q)$, for all $0 \le q \le 1$, implies $f(q, y; x, \beta) \le f(q, y; \bar{x}, \beta)$, for any $0 \le q \le 1$, $y \in R$. Moreover f is pointwise continuous in the $L^1([0, 1], dq)$ norm. In fact, for generic x, $\bar{x}$, we have*

$$|f(q, y; x, \beta) - f(q, y; \bar{x}, \beta)| \le \frac{\beta^2}{2} \int_q^1 |x(q') - \bar{x}(q')|\, dq'.$$

This result is very important. In fact, any functional order parameter can be approximated in the L^1 norm through a piecewise constant one. The pointwise continuity allows us to deal mostly with piecewise constant order parameters.

Now we are ready for the following important definitions.

Definition 1. *The trial auxiliary function, associated to a given mean field spin glass system, as described in Section 2, depending on the functional order parameter x, is defined as*

$$\bar{\alpha}(\beta, h; x) \equiv \log 2 + f(0, h; x, \beta) - \frac{\beta^2}{2} \int_0^1 q\, x(q)\, dq. \tag{14}$$

Notice that in this expression the function f appears evaluated at $q = 0$, and $y = h$, where h is the value of the external magnetic field.

Definition 2. *The Parisi spontaneously broken replica symmetry solution is defined by*

$$\bar{\alpha}(\beta, h) \equiv \inf_x \bar{\alpha}(\beta, h; x), \tag{15}$$

where the infimum is taken with respect to all functional order parameters x.

Of course, by taking the infimum only with respect to replica symmetric order parameters, as in (9), we would get the replica symmetric solution of Sherrington and Kirkpatrick, as exploited for example in the sum rules in [6], and [8].

The main motivation for the introduction of the quantities given by the definitions is the following expected tentative Theorem

Theorem 2. *(***expected***) In the thermodynamic limit, for the partition function defined in (2), we have*

$$\lim_{N\to\infty} N^{-1} E \log Z_N(\beta, h, J) = \bar{\alpha}(\beta, h).$$

Of course, the present technology is far from being able to give a complete rigorous proof. However, in the next Section we will prove that $\bar{\alpha}(\beta, h)$ is at least a rigorous upper bound for $N^{-1} E \log Z_N(\beta, h, J)$, uniformly in N.

4 The main results

The main results of this paper are summarized in the following

Theorem 3. *For all values of the inverse temperature β, and the external magnetic field h, and for any functional order parameter x, the following bound holds*

$$N^{-1}E\log Z_N(\beta,h,J)\le\bar\alpha(\beta,h;x),$$

uniformly in N, where $\bar\alpha(\beta,h;x)$ is defined in (14). Consequently, we have also

$$N^{-1}E\log Z_N(\beta,h,J)\le\bar\alpha(\beta,h),$$

uniformly in N, where $\bar\alpha(\beta,h)$ is defined in (15). Moreover, for the thermodynamic limit, we have

$$\lim_{N\to\infty}N^{-1}E\log Z_N(\beta,h,J)\equiv\alpha(\beta,h)\le\bar\alpha(\beta,h),$$

and

$$\lim_{N\to\infty}N^{-1}\log Z_N(\beta,h,J)\equiv\alpha(\beta,h)\le\bar\alpha(\beta,h),$$

J-almost surely.

The proof is long, and will be split in a series of intermediate results. Consider a generic piecewise constant functional order parameter x, as in (8), and define the auxiliary partition function $\tilde Z$, as follows

$$\begin{aligned}\tilde Z_N(\beta,h;t;x;J)\ \equiv\ &\sum_{\sigma_1\ldots\sigma_N}\exp\Bigg(\beta\sqrt{\frac{t}{N}}\sum_{(i,j)}J_{ij}\sigma_i\sigma_j\\&+\beta h\sum_i\sigma_i+\beta\sqrt{1-t}\sum_{a=1}^{K}\sqrt{q_a-q_{a-1}}\sum_i J_i^a\sigma_i\Bigg)\end{aligned}\tag{16}$$

Here, we have introduced additional independent centered unit Gaussian J_i^a, $a=1,\ldots,K$, $i=1,\ldots,N$. The interpolating parameter t runs in the interval $[0,1]$.

For $a=1,\ldots,K$, let us call E_a the average with respect to all random variables J_i^a, $i=1,\ldots,N$. Analogously, we call E_0 the average with respect to all J_{ij}, and denote by E averages with respect to all J random variables.

Now we define recursively the random variables $Z_0,Z_1,\ldots,Z_K$

$$Z_K=\tilde Z_N(\beta,h;t;x;J),\ Z_{K-1}^{m_K}=E_KZ_K^{m_K},\ldots,\ Z_0^{m_1}=E_1Z_1^{m_1},\tag{17}$$

and the auxiliary function $\tilde\alpha_N(t)$

$$\tilde\alpha_N(t)=\frac{1}{N}E_0\log Z_0.\tag{18}$$

Notice that, due to the partial integrations, any Z_a depends only on the J_{ij}, and on the J_i^b with $b \leq a$, while in $\tilde{\alpha}$ all J noises have been completely averaged out.

The basic motivation for the introduction of $\tilde{\alpha}$ is given by

Lemma 1. *At the extreme values of the interpolating parameter t we have*

$$\tilde{\alpha}_N(1) = \frac{1}{N} E \log Z_N(\beta, h, J), \tag{19}$$

$$\tilde{\alpha}_N(0) = \log 2 + f(0, h; x, \beta), \tag{20}$$

where f is as described in Section 3.

The proof is simple. In fact, at $t = 1$, the J_i^a disappear, and $\tilde{Z}$ reduces to Z in (2). On the other hand, at $t = 0$, the two site couplings J_{ij} disappear, while all effects of the J_i^a factorize with respect to the sites i. Therefore, we are essentially reduced to a one site problem, and it is immediate to see that the averages in (17) reduce to the Gaussian averages necessary for the computation of the solution of the antiparabolic problem (10), (11), as given by the repeated application of (12), with the f function evaluated at $q = 0$, and $y = h$.

It is clear that now we have to proceed to the calculation of the t derivative of $\tilde{\alpha}_N(t)$. But we need some few additional definitions. Introduce the random variables f_a, $a = 1, \ldots, K$,

$$f_a = \frac{Z_a^{m_a}}{E_a(Z_a^{m_a})}, \tag{21}$$

and notice that they depend only on the J_i^b with $b \leq a$, and are normalized, $E(f_a) = 1$. Moreover, we consider the t-dependent state ω associated to the *Boltzmannfaktor* in (16), and its replicated Ω. A very important role is played by the following states $\tilde{\omega}_a$, and their replicated ones $\tilde{\Omega}_a$, $a = 0, \ldots, K$, defined as

$$\tilde{\omega}_K(.) = \omega(.), \;\; \tilde{\omega}_a(.) = E_{a+1} \ldots E_K(f_{a+1} \ldots f_K \omega(.)). \tag{22}$$

Finally, we define the $\langle . \rangle_a$ averages, through a generalization of (5),

$$\langle . \rangle_a = E(f_1 \ldots f_a \tilde{\Omega}_a(.)). \tag{23}$$

As it will be clear in the following, the $\langle . \rangle_a$ averages are able, in a sense, to concentrate the overlap fluctuations around the value q_a.

Now, we have all definitions in order to be able to state the following important results.

Theorem 4. *The t derivative of $\tilde{\alpha}_N(t)$ in (18) is given by*

$$\begin{aligned}\frac{d}{dt}\tilde{\alpha}_N(t) &= -\frac{\beta^2}{4}(1-\sum_{a=0}^{K}(m_{a+1}-m_a)q_a^2) \\ &\quad -\frac{\beta^2}{4}\sum_{a=0}^{K}(m_{a+1}-m_a)\langle(q_{12}-q_a)^2\rangle_a. \qquad (24)\end{aligned}$$

Theorem 5. *For any functional order parameter, of the type given in (8), the following sum rule holds*

$$\bar{\alpha}(\beta,h;x)=\frac{1}{N}E\log Z_N(\beta,h;J)+\frac{\beta^2}{4}\sum_{a=0}^{K}(m_{a+1}-m_a)\int_0^1\langle(q_{12}-q_a)^2\rangle_a(t)\,dt. \qquad (25)$$

Clearly, Theorem 5 follows from the previous Theorem 4, by integrating with respect to t, taking into account the boundary values in Lemma 1, and the definition of $\bar{\alpha}(\beta,h;x)$ given in Section 3. Moreover, one should use also the obvious identity

$$\frac{1}{2}(1-\sum_{a=0}^{K}(m_{a+1}-m_a)q_a^2)=\int_0^1 q\,x(q)\,dq. \qquad (26)$$

By taking into account that all terms in the sum rule are nonnegative, since $m_{a+1}\geq m_a$, we can immediately establish the validity of Theorem 3.

Now we must attack Theorem 4. The proof is straightforward, and involves integration by parts with respect to the external noises. We only sketch the main points. Let us begin with

Lemma 2. *The t derivative of $\tilde{\alpha}_N(t)$ in (18) is given by*

$$\frac{d}{dt}\tilde{\alpha}_N(t)=\frac{1}{N}E(f_1f_2\dots f_K Z_K^{-1}\partial_t Z_K), \qquad (27)$$

where

$$\begin{aligned}Z_K^{-1}\partial_t Z_K &= \tilde{Z}_N^{-1}\partial_t\tilde{Z}_N \\ &= \frac{\beta}{2\sqrt{tN}}\sum_{(ij)}J_{ij}\omega(\sigma_i\sigma_j)-\frac{\beta}{2\sqrt{1-t}}\sum_{a=1}^{K}\sqrt{q_a-q_{a-1}}\sum_i J_i^a\omega(\sigma_i).\end{aligned}$$

The proof is very simple. In fact, from the definitions in (17), we have, for $a = 0, 1, \dots, K-1$,

$$Z_a^{-1}\partial_t Z_a = E_{a+1}(f_{a+1} Z_{a+1}^{-1}\partial_t Z_{a+1}). \tag{28}$$

The rest follows from iteration of this formula, and simple calculations.

Clearly, now we have to evaluate

$$\begin{aligned} E(J_{ij} f_1 f_2 \dots f_K \omega(\sigma_i\sigma_j)) &= \sum_{a=1}^{K} E(\dots \partial_{J_{ij}} f_a \dots \omega(\sigma_i\sigma_j)) + E(f_1 \dots f_K \partial_{J_{ij}} \omega(\sigma_i\sigma_j)), \\ E(J_i^a f_1 f_2 \dots f_K \omega(\sigma_i)) &= \sum_{b=1}^{K} E(\dots \partial_{J_i^a} f_b \dots \omega(\sigma_i)) + E(f_1 \dots f_K \partial_{J_i^a} \omega(\sigma_i)), \end{aligned}$$

where we have exploited standard integration by parts on the Gaussian J variables.

The following lemma gives all additional information necessary for the proof of Theorem 4.

Lemma 3. *For the J-derivatives we have*

$$\begin{aligned} \partial_{J_{ij}} \omega(\sigma_i\sigma_j) &= \beta\sqrt{\frac{t}{N}}(1 - \omega^2(\sigma_i\sigma_j)), & (29) \\ \partial_{J_i^a} \omega(\sigma_i) &= \beta\sqrt{1-t}\sqrt{q_a - q_{a-1}}(1 - \omega^2(\sigma_i)), & (30) \\ \partial_{J_{ij}} f_a &= \beta\sqrt{\frac{t}{N}} m_a f_a (\tilde{\omega}_a(\sigma_i\sigma_j) - \tilde{\omega}_{a-1}(\sigma_i\sigma_j)), & (31) \\ \partial_{J_i^a} f_b &= 0, \ \textit{if } b < a, & (32) \\ \partial_{J_i^a} f_a &= \beta\sqrt{1-t}\sqrt{q_a - q_{a-1}} m_a f_a \tilde{\omega}_a(\sigma_i), & (33) \\ \partial_{J_i^a} f_b &= \beta\sqrt{1-t}\sqrt{q_a - q_{a-1}} m_b f_b (\tilde{\omega}_b(\sigma_i) - \tilde{\omega}_{b-1}(\sigma_i)), \ \textit{if } b > a & (34) \end{aligned}$$

The proof of (29) and (30) is a standard calculation. On the other hand, Eq. (31) follows from the definition (21) and the easily established

$$\begin{aligned} \partial_{J_{ij}} Z_a^{m_a} &= m_a Z_a^{m_a} Z_a^{-1} \partial_{J_{ij}} Z_a, \\ Z_a^{-1}\partial_{J_{ij}} Z_a &= E_{a+1}(f_{a+1} Z_{a+1}^{-1} \partial_{J_{ij}} Z_{a+1}), \ a = 1, \dots, K-1, \\ Z_K^{-1}\partial_{J_{ij}} Z_K &= \tilde{Z}_N^{-1}\partial_{J_{ij}} \tilde{Z}_N = \beta\sqrt{\frac{t}{N}}\omega(\sigma_i\sigma_j), \\ Z_a^{-1}\partial_{J_{ij}} Z_a &= \beta\sqrt{\frac{t}{N}} E_{a+1}(f_{a+1} \dots f_K \omega(\sigma_i\sigma_j)) = \beta\sqrt{\frac{t}{N}}\tilde{\omega}_a(\sigma_i\sigma_j). \end{aligned}$$

In the same way, we can establish (32), (33), (34). But here we have to take into account that Z_b does not depend on J_i^a if $b < a$.

A careful combination of all information given by Lemma 2 and Lemma 3, finally leads to the proof of Theorem 4. On the other hand, the main Theorem 3 follows easily from Theorem 5, and the results of [9].

5 Broken replica symmetry bound for the ground state energy

Let us consider the ground state energy density $-e_N(J,h)$ defined as

$$-e_N(J,h) \equiv \frac{1}{N}\inf_{\sigma} H_N(\sigma,h,J) = -\lim_{\beta\to\infty}\frac{\ln Z_N(\beta,h,J)}{\beta N}. \tag{35}$$

By taking the expectation values we also have

$$e_N(h) \equiv E(e_N(J,h)) = \lim_{\beta\to\infty}\frac{\alpha_N(\beta,h)}{\beta}. \tag{36}$$

From the results of the previous Section, we have, for any functional order parameter x,

$$\frac{E(\ln Z_N(\beta,h,J))}{\beta N} \leq \beta^{-1}\bar{\alpha}(\beta,h;x), \tag{37}$$

uniformly in N.

Let us now introduce an arbitrary sequence

$$0 \leq \bar{m}_1 \leq \bar{m}_2 \leq \ldots \leq \bar{m}_K, \tag{38}$$

and the corresponding order parameter $\bar{x}$, defined as in (8), but with all m_a replaced by $\bar{m}_a$. Notice that there is no upper bound equal to 1 for $\bar{m}_K$, and consequently for $\bar{x}$. However, for sufficiently large β, we definitely have $\bar{m}_K \leq \beta$. Therefore, we can take in (37) the order parameter x defined by $x(q) = \bar{x}(q)/\beta$, with $0 \leq x(q) \leq 1$. Then we can easily establish the following Lemma.

Lemma 4. *In the limit $\beta \to \infty$ we have*

$$\lim_{\beta\to\infty}\beta^{-1}\bar{\alpha}(\beta,h;x) = \tilde{\alpha}_N(h;\bar{x}) \equiv \bar{f}(0,h;\bar{x}) - \frac{1}{2}\int_0^1 q\bar{x}(q)\,dq, \tag{39}$$

where the function $\bar{f}$, with values $\bar{f}(q,y;\bar{x})$ satisfies the antiparabolic equation

$$(\partial_q\bar{f})(q,y) + \frac{1}{2}(\bar{f}''(q,y) + \bar{x}(q)\bar{f}'^2(q,y)) = 0, \tag{40}$$

with final condition

$$\bar{f}(1,y) = |y|. \tag{41}$$

The proof is easy. In fact, the recursive solution for f coming from (12), allows to prove immediately

$$\lim_{\beta\to\infty} \beta^{-1} f(q,y;\bar{x}/\beta) = \bar{f}(q,y;\bar{x}), \tag{42}$$

by taking into account the elementary $\lim_{\beta\to\infty} \beta^{-1} \log\cosh(\beta y) = |y|$.

Therefore we have established

Theorem 6. *The following inequalities hold*

$$\begin{aligned} e_N(h) &\leq \tilde{\alpha}_N(h;\bar{x}), & (43)\\ e_N(h) &\leq \tilde{\alpha}_N(h) \equiv \inf_{\bar{x}} \tilde{\alpha}_N(h;\bar{x}), & (44)\\ \lim_{N\to\infty} e_N(h) \equiv e_0(h) &\leq \tilde{\alpha}_N(h;\bar{x}), & (45)\\ e_0(h) &\leq \tilde{\alpha}_N(h). & (46) \end{aligned}$$

A detailed study of the numerical information coming from the variational bound of Theorem 6 will be presented in a forthcoming paper [11].

6 Broken replica symmetry bounds in the p-spin model

The methods developed in the previous Sections can be easily extended to the Derrida p-spin model [2], [4], [3], [17]. We give here only a brief sketch. A more detailed treatment will be presented elsewhere [10].

Now the Hamiltonian contains a term coupling each group made of p spins

$$H_N(\sigma,h,J) = -(\frac{p!}{2N^{p-1}})^{\frac{1}{2}} \sum_{(i_1,\dots i_p)} J_{i_1\dots i_p}\sigma_{i_1}\dots\sigma_{i_p} - h\sum_i \sigma_i. \tag{47}$$

For the sake of simplicity, in the following we consider only the case of even p. Piecewise constant order parameters are introduced as in (7), (8), where now we assume $q_K = p/2$. We still introduce the function f, defined by (10), with $0 \leq q \leq p/2$, and final condition

$$f(p/2,y) = \log\cosh(\beta y). \tag{48}$$

We also introduce the change of variables $q \to \bar{q}$, defined by $2q = p\bar{q}^{p-1}$, so that, in particular, $\bar{q}_K \leq 1$. The definitions (14) and (15) must be modified as follows.

Definition 3. *The trial auxiliary function, associated to a given p-spin mean field spin glass system, as described before, depending on the functional order parameter x, is defined as*

$$\bar{\alpha}(\beta, h; x) \equiv \log 2 + f(0, h; x, \beta) - \frac{\beta^2}{2} \int_0^{\frac{p}{2}} \bar{q}(q)\, x(q)\, dq. \tag{49}$$

Definition 4. *The spontaneously broken replica symmetry solution for the p-spin model is defined by*

$$\bar{\alpha}(\beta, h) \equiv \inf_x \bar{\alpha}(\beta, h; x), \tag{50}$$

where the infimum is taken with respect to all functional order parameters x.

With the same procedure as described in Section 4, we arrive to the sum rule given by

Theorem 7. *In the p-spin model, for any functional order parameter, the following sum rule holds*

$$\begin{aligned} \bar{\alpha}(\beta, h; x) &= \frac{1}{N} E \log Z_N(\beta, h; J) \\ &+ \frac{\beta^2}{4} \sum_{a=0}^{K} (m_{a+1} - m_a) \int_0^1 \langle q_{12}^p - p q_{12} \bar{q}_a^{p-1} + (p-1) \bar{q}_a^p \rangle_a(t)\, dt \\ &+ O(\frac{1}{N}), \end{aligned} \tag{51}$$

where $\bar{\alpha}(\beta, h; x)$ is defined in(49).

Notice that the terms under the sum are still positive. The $O(\frac{1}{N})$ correction is typical of the p-spin models.

From the sum rule we have also

Theorem 8. *In the p-spin model, for any functional order parameter x, the following bound holds*

$$N^{-1} E \log Z_N(\beta, h, J) \leq \bar{\alpha}(\beta, h; x) + O(\frac{1}{N}),$$

where $\bar{\alpha}(\beta, h; x)$ is defined in (49). Consequently, we have also

$$N^{-1} E \log Z_N(\beta, h, J) \leq \bar{\alpha}(\beta, h) + O(\frac{1}{N}),$$

where $\bar{\alpha}(\beta, h)$ is defined in (50). Moreover, for the thermodynamic limit, we have

$$\lim_{N\to\infty} N^{-1} E \log Z_N(\beta, h, J) \equiv \alpha(\beta, h) \leq \bar{\alpha}(\beta, h),$$

and

$$\lim_{N\to\infty} N^{-1} \log Z_N(\beta, h, J) \equiv \alpha(\beta, h) \leq \bar{\alpha}(\beta, h),$$

J-almost surely.

We refer to [10] for a more detailed treatment.

7 Conclusions and outlook for future developments

Without exploiting any reference to the zero replica trick, or to the cavity method, we have found a way to prove that the true free energy for the mean field spin glass model is bounded below by its spontaneously broken symmetry expression, given in the frame of the Parisi *Ansatz*. The method extends easily to the Derrida p-spin model. The key role is played by the auxiliary function $\tilde{\alpha}_N(t)$, defined in (18). Our method, in its very essence, is a generalization of the mechanical analogy introduced in [6], for the comparison with the replica symmetric approximation.

The main open problems are given by the extension of these methods to other disordered systems, as for example the mean field neural network models. Moreover, the sum rules developed here could be taken as the starting point to prove that the additional positive terms do really vanish in the infinite volume limit. This would prove rigorously the validity of the broken replica *Ansatz*.

We plan to report on these problems in future papers.

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